DAY TWENTY NINE

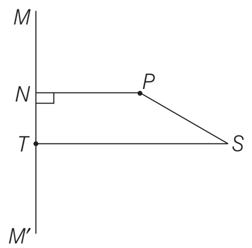
Hyperbola

Learning & Revision for the Day

- Concept of Hyperbola
- Equations of Hyperbola in Standard Form
- Tangent to a Hyperbola
- Normal to a Hyperbola
- Pole and Polar
- Director Circle
- Asymptotes
- Rectangular or Equilateral Hyperbola

Concept of Hyperbola

Hyperbola is the locus of a point in a plane which moves in such a way that the ratio of its distance from a fixed point (focus) in the same plane to its distance from a fixed line (directrix) is always constant which is always greater than unity.



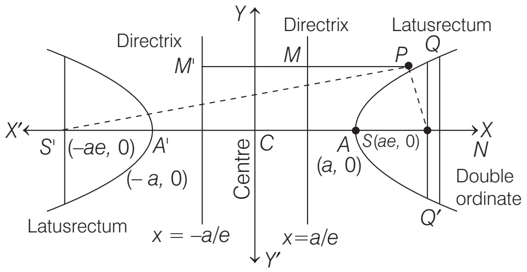
Mathematically, $\frac{SP}{PN} = e$, where $e > 1$.

Terms Related to Hyperbola

Some important terms related to hyperbola are given below.

1. **Vertices** The points A and A', where the curve meets the line joining the foci S and S', are called the vertices of the hyperbola.
2. **Transverse and conjugate axes** Transverse axis is the one which lie along the line passing through the foci and perpendicular to the directrices and conjugate axis and conjugate axis is the one which is perpendicular to the transverse axis and passes through the mid-point of the foci i.e. centre.
3. **Centre** The mid-point C of AA' bisects every chord of the hyperbola passing through it and is called the centre of the hyperbola.
4. **Focal chord** A chord of a hyperbola which is passing through the focus is called a focal chord of the hyperbola.
5. **Directrix** A line which is perpendicular to the axis and it lies between centre and vertex. The equation of directrix is $x = \pm \frac{a}{e}$.
6. **Double ordinates** If Q be a point on the hyperbola draw QN perpendicular to the axis of the hyperbola and produced to meet the curve again at Q'. Then, QQ' is called a double ordinate of Q.

7. **Latusrectum** The double ordinate passing through focus is called latusrectum.



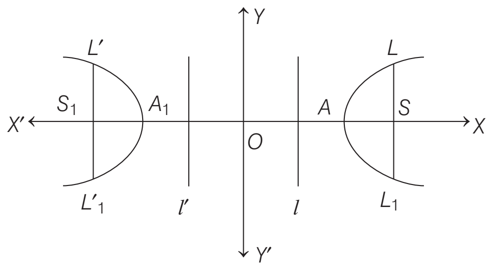
NOTE
- The vertex divides the join of focus and the point of intersection of directrix with axis internally and externally in the ratio e : 1.
- Domain and range of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $x \le -a$ or $x \ge a$ and $y \in R$, respectively.
- The line through the foci of the hyperbola is called its transverse axis.
- The line through the centre and perpendicular to the transverse axis of the hyperbola is called its conjugate axis.

Equations of Hyperbola in Standard Form

If the centre of the hyperbola is at the origin and foci are on the X-axis or Y-axis, then that types of equation are called standard equation of an ellipse.

1. Hyperbola of the Form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

When the hyperbola is in the given form, then it is also called the equation of auxiliary circle.



(i) Centre, $O(0, 0)$

(ii) Foci : $S(ae, 0), S_1(-ae, 0)$

(iii) Vertices : $A(a, 0)$, $A_1(-a, 0)$

(iv) Directrices $l : x = \frac{a}{e}, l' : x = -\frac{a}{e}$

(v) Length of latusrectum, $LL_1 = L'L'_1 = \frac{2b^2}{a}$

(vi) Eccentricity, $e = \sqrt{1+\left(\frac{b}{a}\right)^2}$ or $b^2 = a^2(e^2-1)$

(vii) Length of transverse axis, $2a$

(viii) Length of conjugate axis, $2b$

(ix) Equation of transverse axis, $y = 0$

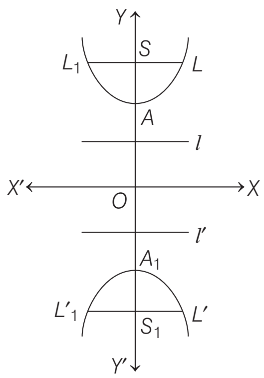
(x) Equation of conjugate axis, $x = 0$

(xi) Focal distances of a point on the hyperbola is $ex \pm a$.

(xii) Difference of the focal distances of a point on the hyperbola is $2a$.

2. Conjugate Hyperbola $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola. The conjugate hyperbola of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



(i) Centre, $O(0, 0)$

(ii) Foci, $S(0, be), S_1(0, -be)$

(iii) Vertices, $A(0, b), A_1(0, -b)$

(iv) Directrices $l : y = \frac{b}{e}, l' : y = -\frac{b}{e}$

(v) Length of latusrectum $LL_1 = L'L_1' = \frac{2a^2}{b}$

(vi) Eccentricity, $e = \sqrt{1+\left(\frac{a}{b}\right)^2}$ as $a^2 = b^2(e^2-1)$

(vii) Length of transverse axis, $2b$

(viii) Length of conjugate axis, $2a$

(ix) Equation of transverse axis, $x = 0$

(x) Equation of conjugate axis, $y = 0$

(xi) Focal distances of a point on the hyperbola is $ey \pm b$.

(xii) Difference of the focal distances of a point on the hyperbola is $2b$.

NOTE If the centre of the hyperbola is (h, k) and axes are parallel to the coordinate axes, then its equation is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.

Results on Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

1. The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.
2. The points of intersection of the directrix with the transverse axis are known as foot of the directrix.
3. Latusrectum $(l)=2e$ (distance between the focus and the foot of the corresponding directrix).
4. The parametric equation of a hyperbola is $x=a\sec\theta$ and $y=b\tan\theta$, where $\theta\in[0,2\pi)$.
5. The position of a point (h,k) with respect to the hyperbola S lie inside, on or outside the hyperbola, if

$S_1>0,\ S_1=0$ or $S_1<0$ where, $S_1\equiv\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$

Tangent to a Hyperbola

A line which intersects the hyperbola at only one point is called the tangent to the hyperbola.

(i) In point (x_1,y_1) form, $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$

(ii) In slope 'm' form, $y=mx\pm\sqrt{a^2m^2-b^2}$

(iii) In parametric form, $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$ at $(a\sec\theta,\ b\tan\theta)$.

(iv) The line $y=mx+c$ touches the hyperbola, iff $c^2=a^2m^2-b^2$ and the point of contact is $\left(\pm\frac{a^2m}{c},\pm\frac{b^2}{c}\right)$, where $c=\sqrt{a^2m^2-b^2}$.

Results on Tangent

1. Two tangents can be drawn from a point to a hyperbola.
2. The point of intersection of tangents at t_1 and t_2 to the curve $xy=c^2$ is $\left(\frac{2c\,t_1t_2}{t_1+t_2},\frac{2c}{t_1+t_2}\right)$.
3. The tangent at the point $P(a\sec\theta_1,b\tan\theta_1)$ and $Q(a\sec\theta_2,b\tan\theta_2)$ intersect at the point
$$R\left(\frac{a\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)},\frac{b\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}\right).$$
4. The equation of pair of tangents drawn from an external point $P(x_1,y_1)$ to the hyperbola is $SS_1=T^2$

where $S=\frac{x^2}{a^2}-\frac{y^2}{b^2}-1,\ S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$

and $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

5. The equation of chord of contact is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ or $T=0$.
6. The equation of chord of the hyperbola, whose mid-point is (x_1,y_1) is $T=S_1$ i.e. $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$.
7. Equation of chord joining the points $(a\sec\alpha,b\tan\alpha)$ and $(a\sec\beta,b\tan\beta)$ is
$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right)-\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha+\beta}{2}\right).$$

Normal to a Hyperbola

A line which is perpendicular to the tangent of the hyperbola is called the normal to the hyperbola.

(i) In point (x_1,y_1) form $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$.

(ii) In slope 'm' form $y=mx\mp\frac{m(a^2+b^2)}{\sqrt{a^2-m^2b^2}}$ and the point of intersection is $\left(\pm\frac{a^2}{\sqrt{a^2-b^2m^2}},\mp\frac{b^2m}{\sqrt{a^2-b^2m^2}}\right)$

(iii) In parametric form, $ax\cos\theta+by\cot\theta=a^2+b^2$ at $(a\sec\theta,b\tan\theta)$

Results on Normals

1. If the straight line $lx+my+n=0$ is a normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $\frac{a^2}{l^2}-\frac{b^2}{m^2}=\frac{(a^2+b^2)^2}{n^2}$
2. Four normals can be drawn from any point to a hyperbola.
3. The line $y=mx+c$ will be a normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, if $c^2=\frac{m^2(a^2+b^2)^2}{a^2-b^2m^2}$.
4. If the normals at four points $P(x_1,y_1),Q(x_2,y_2),R(x_3,y_3)$ and $S(x_4,y_4)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are concurrent, then
$$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4.$$

Pole and Polar

Let P be a point inside or outside a hyperbola. Then, the locus of the point of intersection of two tangents to the hyperbola at the point where secants drawn through P intersect the hyperbola, is called the **polar** of point P with respect to the hyperbola and the point P is called the **pole** of the polar.

(i) Polars of two points $P(x_1 y_1)$ and $Q(x_2, y_2)$ of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ and $\frac{xx_2}{a^2} - \frac{yy_2}{b^2} = 1$

(ii) If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then $\frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2} = 1$ and the polar of $Q(x_2, y_2)$ passes through $P(x_1, y_1)$, then $\frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2} = 1$

Conjugate Points and Conjugate Lines

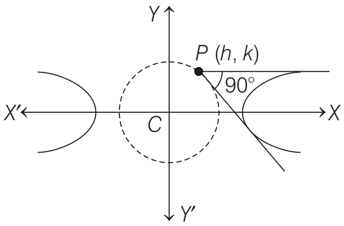
(i) Two points are said to be conjugate points with respect to a hyperbola, if each lies on the polar of the other.

(ii) Two lines are said to be conjugate lines with respect to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if each passes through the pole of the other.

(iii) Two lines $l_1 x + m_1 y + n_1 = 0$ and $l_2 x + m_2 y + n_2 = 0$ are conjugate lines with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $a^2 l_1 l_2 - b^2 m_1 m_2 = n_1 n_2$.

Director Circle

The locus of the point of intersection of the tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, which are perpendicular to each other, is called a director circle.



The equation of director circle is $x^2 + y^2 = a^2 - b^2$.

The circle $x^2 + y^2 = a^2$ is known the auxiliary circle of both hyperbola.

Asymptotes

An asymptote to a curve is a straight line, which touches on it two points at infinity but which itself does not lie entirely at infinity.

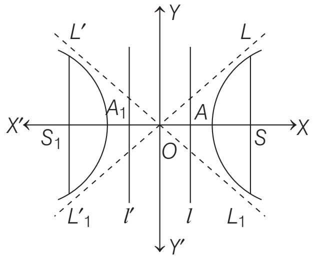
Results on Asymptotes

1. A hyperbola and its conjugate hyperbola have the same asymptotes.
2. The angle between the asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $2\tan^{-1}\left(\frac{b}{a}\right)$.
3. Asymptotes always passes through the centre of the hyperbola.
4. The bisectors of the angle between the asymptotes are the coordinate axes.
5. The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extrimities of each axis parallel to the other axis.

Rectangular or Equilateral Hyperbola

A hyperbola whose asymptotes include a right angle is said to be rectangular hyperbola or if the lengths of transverse and conjugate axes of any hyperbola is equal, then it is said to be a rectangular hyperbola.

Rectangular Hyperbola of the Form $x^2 - y^2 = a^2$



1. Asymptotes are perpendicular lines i.e. $x \pm y = 0$
2. Eccentricity, $e = \sqrt{2}$,
3. Centre, $O(0, 0)$
4. Foci, S and S_1 $(\pm\sqrt{2}\,a, 0)$
5. Directrices, $x = \pm\frac{a}{\sqrt{2}}$
6. Latusrectum $= 2a$
7. Point form, $x = x_1, y = y_1$

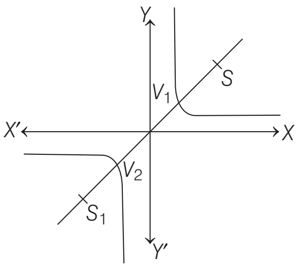
   Equation of tangent, $xx_1 - yy_1 = a^2$

   Equation of normal, $\frac{x_1}{x} + \frac{y_1}{y} = 2$
8. Parametric form, $x = a\sec\theta, y = a\tan\theta$

   Equation of tangent, $x\sec\theta - y\tan\theta = a$

   Equation of normal, $\frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 2a$

Rectangular Hyperbola of the Form $xy = c^2$



1. Asymptotes are perpendicular lines i.e. $x = 0$ and $y = 0$
2. Eccentricity, $e = \sqrt{2}$
3. Centre, $(0, 0)$
4. Foci

   $S(\sqrt{2}c, \sqrt{2}c), S_1(-\sqrt{2}c, -\sqrt{2}c)$
5. Vertices, $V_1(c, c), V_2(-c, -c)$
6. Directrices, $x + y = \pm\sqrt{2}\,c$
7. Latusrectum $= 2\sqrt{2}\,c$
8. Point form, $x = x_1, y = y_1$

   Equation of tangent, $xy_1 + yx_1 = 2c^2$

   $\Rightarrow \quad \frac{x}{x_1} + \frac{y}{y_1} = 2$

   Equation of Normal, $xx_1 - yy_1 = x_1^2 - y_1^2$
9. Parametric form : $x = ct, y = \frac{c}{t}$

   Equation of tangent, $x + yt^2 = 2ct$

   Equation of normal, $t^2 x - y = c\left(t^3 - \frac{1}{t}\right)$

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 If the vertices of hyperbola are $(0, \pm 6)$ and its eccentricity is $\frac{5}{3}$, then → **NCERT Exemplar**

I. The equation of hyperbola is $\frac{y^2}{36} - \frac{x^2}{64} = 1$.

II. The foci of hyperbola are $(0, \pm 10)$.

(a) Both I and II are true (b) Only I is true
(c) Only II is true (d) Both I and II are false

2 The equation of the hyperbola, the lenght of whose latusrectum is 8 and eccentricity is $\frac{3}{\sqrt{5}}$, is

(a) $5x^2 - 4y^2 = 100$ (b) $4x^2 - 5y^2 = 100$
(c) $-4x^2 + 5y^2 = 100$ (d) $-5x^2 + 4y^2 = 100$

3 The eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which passes through the points $(3, 0)$ and $(3\sqrt{2}, 2)$, is → **NCERT Exemplar**

(a) $\frac{13}{3}$ (b) $\sqrt{\frac{13}{3}}$ (c) $\frac{\sqrt{13}}{9}$ (d) $\frac{\sqrt{13}}{3}$

4 Eccentricity of hyperbola $\frac{x^2}{k} + \frac{y^2}{k^2} = 1 \, (k < 0)$ is

(a) $\sqrt{1+k}$ (b) $\sqrt{1-k}$ (c) $\sqrt{1+\frac{1}{k}}$ (d) $\sqrt{1-\frac{1}{k}}$

5 The equation of the hyperbola whose foci are $(-2, 0)$ and $(2, 0)$ and eccentricity is 2, is given by → **AIEEE 2011**

(a) $-3x^2 + y^2 = 3$ (b) $x^2 - 3y^2 = 3$
(c) $3x^2 - y^2 = 3$ (d) $-x^2 + 3y^2 = 3$

6 The length of transverse axis of the hyperbola $3x^2 - 4y^2 = 32$ is

(a) $\frac{8\sqrt{2}}{\sqrt{3}}$ (b) $\frac{16\sqrt{2}}{\sqrt{3}}$ (c) $\frac{3}{32}$ (d) $\frac{64}{3}$

7 The difference between the length $2a$ of the transverse axis of a hyperbola of eccentricity e and the length of its latusrectum is

(a) $2a(3 - e^2)$ (b) $2a|2 - e^2|$
(c) $2a(e^2 - 1)$ (d) $a(2e^2 - 1)$

8 If eccentricity of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is e and e' is the eccentricity of its conjugate hyperbola, then

(a) $e = e'$ (b) $ee' = 1$
(c) $\frac{1}{e^2} + \frac{1}{(e')^2}$ (d) None of these

9 A hyperbola, having the transverse axis of length $2\sin\theta$, is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then, its equation is

(a) $x^2 \operatorname{cosec}^2\theta - y^2 \sec^2\theta = 1$ (b) $x^2 \sec^2\theta - y^2 \operatorname{cosec}^2\theta = 1$
(c) $x^2 \sin^2\theta - y^2 \cos^2\theta = 1$ (d) $x^2 \cos^2\theta - y^2 \sin^2\theta = 1$

10 Length of the latusrectum of the hyperbola $xy - 3x - 4y + 8 = 0$ is

(a) 4 (b) $4\sqrt{2}$ (c) 8 (d) None of these

11 The eccentricity of the hyperbola whose latusrectum is 8 and conjugate axis is equal to half of the distance between the foci is → **NCERT Exemplar**

(a) $\frac{4}{3}$ (b) $\frac{4}{\sqrt{3}}$ (c) $\frac{2}{\sqrt{3}}$ (d) None of these

12 A general point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

(a) $(a\sin\theta, b\cos\theta)$ (where, θ is parameter)
(b) $(a\tan\theta, b\sec\theta)$ (where, θ is parameter)
(c) $\left(a\frac{e^t + e^{-t}}{2}, b\frac{e^t - e^{-t}}{2}\right)$ (where, t is parameter)
(d) None of the above

13 For the hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$. Which of the following remains constant when α varies?

(a) Directrix (b) Abscissae of vertices
(c) Abscissae of foci (d) Eccentricities

14 The product of the perpendicular from two foci on any tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is

(a) a^2 (b) b^2 (c) $-a^2$ (d) $-b^2$

15 If a line $21x + 5y = 116$ is tangent to the hyperbola $7x^2 - 5y^2 = 232$, then point of contact is

(a) $(-6, 3)$ (b) $(6, -2)$ (c) $(8, 2)$ (d) None of these

16 If e_1 is the eccentricity of the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ and e_2 is the eccentricity of the hyperbola passing through the foci of the ellipse and $e_1 e_2 = 1$, then equation of the hyperbola is

(a) $\frac{x^2}{9} - \frac{y^2}{16} = 1$ (b) $\frac{x^2}{16} - \frac{y^2}{9} = -1$
(c) $\frac{x^2}{9} - \frac{y^2}{25} = 1$ (d) None of these

17 If the line $2x + \sqrt{6}y = 2$ touches the hyperbola $x^2 - 2y^2 = 4$, then the point of contact is

(a) $(-2, \sqrt{6})$ (b) $(-5, 2\sqrt{6})$ (c) $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$ (d) $(4, -\sqrt{6})$

18 If a hyperbola passes through the point $P(\sqrt{2}, \sqrt{3})$ and has foci at $(\pm 2, 0)$, then the tangent to this hyperbola at P also passes through the point → **JEE Mains 2017**

(a) $(3\sqrt{2}, 2\sqrt{3})$ (b) $(2\sqrt{2}, 3\sqrt{3})$ (c) $(\sqrt{3}, \sqrt{2})$ (d) $(-\sqrt{2}, -\sqrt{3})$

19 The common tangent to $9x^2 - 4y^2 = 36$ and $x^2 + y^2 = 3$ is

(a) $y - 2\sqrt{3}x - \sqrt{39} = 0$ (b) $y + 2\sqrt{3}x + \sqrt{39} = 0$
(c) $y - 2\sqrt{3}x + \sqrt{39} = 0$ (d) None of the above

20 The locus of a point $P(\alpha, \beta)$ moving under the condition that the line $y = \alpha x + \beta$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is **➔ AIEEE 2005**

(a) a hyperbola (b) a parabola (c) a circle (d) an ellipse

21 The straight line $x + y = \sqrt{2}p$ will touch the hyperbola $4x^2 - 9y^2 = 36$, if

(a) $p^2 = 2$ (b) $p^2 = 5$ (c) $5p^2 = 2$ (d) $2p^2 = 5$

22 The locus of the point of intersection of perpendicular tangents to the hyperbola $\frac{x^2}{3} - \frac{y^2}{1} = 1$ is

(a) $x^2 + y^2 = 2$ (b) $x^2 + y^2 = 3$
(c) $x^2 - y^2 = 3$ (d) $x^2 + y^2 = 4$

23 If P is a point on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ and N is foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T. If O is the centre of the hyperbola, then $OT \cdot ON$ is equal to

(a) 9 (b) 4 (c) e^2 (d) None of these

24 The locus of the points of intersection of perpendicular tangents to $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is

(a) $x^2 - y^2 = 7$ (b) $x^2 - y^2 = 25$
(c) $x^2 + y^2 = 25$ (d) $x^2 + y^2 = 7$

25 Tangents are drawn from points on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ to the circle $x^2 + y^2 = 9$. The locus of the mid-point of the chord of contact is

(a) $x^2 + y^2 = \frac{x^2}{9} - \frac{y^2}{4}$ (b) $(x^2 + y^2)^2 = \frac{x^2}{9} - \frac{y^2}{4}$
(c) $(x^2 + y^2)^2 = 81\left(\frac{x^2}{9} - \frac{y^2}{4}\right)$ (d) $(x^2 + y^2)^2 = 9\left(\frac{x^2}{9} - \frac{y^2}{4}\right)$

26 A tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$ meets X-axis at P and Y-axis at Q. Lines PR and QR are drawn such that $OPRQ$ is a rectangle (where, O is the origin). Then, R lies on **➔ JEE Mains 2013**

(a) $\frac{4}{x^2} + \frac{2}{y^2} = 1$ (b) $\frac{2}{x^2} - \frac{4}{y^2} = 1$
(c) $\frac{2}{x^2} + \frac{4}{y^2} = 1$ (d) $\frac{4}{x^2} - \frac{2}{y^2} = 1$

27 Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$ with vertex at the point A. Let B be one of the end points of its latusrectum. If C is the focus of the hyperbola nearest to the point A, then the area of the ΔABC is

(a) $\left(1 - \sqrt{\frac{2}{3}}\right)$ sq unit (b) $\left(\sqrt{\frac{3}{2}} - 1\right)$ sq unit
(c) $\left(1 + \sqrt{\frac{2}{3}}\right)$ sq units (d) $\left(\sqrt{\frac{3}{2}} + 1\right)$ sq units

28 Given the base BC of ΔABC and if $\angle B - \angle C = K$, a constant, then locus of the vertex A is a hyperbola.

(a) No (b) Yes (c) Both (d) None

29 Let $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$, where $\theta + \phi = \frac{\pi}{2}$, be two points on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 13$. If (h, k) is the point of intersection of normals at P and Q, then k is

(a) $\frac{a^2 + b^2}{a}$ (b) $-\frac{(a^2 + b^2)}{a}$
(c) $\frac{a^2 + b^2}{b}$ (d) $-\frac{(a^2 + b^2)}{b}$

30 If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the hyperbola $4x^2 - 9y^2 - 36 = 0$ are at right angles, then $\frac{x_1 x_2}{y_1 y_2}$ is equal to

(a) $\frac{9}{4}$ (b) $-\frac{9}{4}$ (c) $\frac{81}{16}$ (d) $-\frac{81}{16}$

31 If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is

(a) $9x^2 - 8y^2 + 18x - 9 = 0$ (b) $9x^2 - 8y^2 - 18x + 9 = 0$
(c) $9x^2 - 8y^2 - 18x - 9 = 0$ (d) $9x^2 - 8y^2 + 18x + 9 = 0$

32 If chords of the hyperbola $x^2 - y^2 = a^2$ touch the parabola $y^2 = 4ax$. Then, the locus of the middle points of these chords is

(a) $y^2 = (x - a)x^3$ (b) $y^2(x - a) = x^3$
(c) $x^2(x - a) = x^3$ (d) None of these

33 The locus of middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$ is

(a) $3x - 4y = 4$ (b) $3y - 4x + 4 = 0$
(c) $4x - 3y = 3$ (d) $3x - 4y = 2$

34 The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is **➔ AIEEE 2002**

(a) $\frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$ (b) $\frac{x}{x_1 - x_2} + \frac{y}{y_1 - y_2} = 1$
(c) $\frac{x}{y_1 + y_2} + \frac{y}{x_1 + x_2} = 1$ (d) $\frac{x}{y_1 - y_2} + \frac{y}{x_1 - x_2} = 1$

35 Match the vertices (v) and foci (f) of hyperbola given in Column I with their corresponding equation given in Column II and choose the correct option from the codes given below.

Column I		Column II	
A.	$v(\pm 2, 0), f(\pm 3, 0)$	1.	$\frac{y^2}{25} - \frac{x^2}{39} = 1$
B.	$v(0, \pm 5), f(0, \pm 8)$	2.	$\frac{y^2}{9} - \frac{x^2}{16} = 1$
C.	$v(0, \pm 3), f(0 \pm 5)$	3.	$\frac{x^2}{4} - \frac{y^2}{5} = 1$

Codes

	A	B	C		A	B	C
(a)	3	1	2	(b)	1	3	2
(c)	3	2	1	(d)	2	1	3

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 If chords of the hyperbola $x^2 - y^2 = a^2$ touch the parabola $y^2 = 4ax$. Then, the locus of their middle point is

(a) $y^2(x-a) = 2x^2$ (b) $y^2(x-a) = x^3$
(c) $y^2(x-a) = x^4$ (d) $y^2(x+a) = x^3$

2 The equation of a tangent to the hyperbola $3x^2 - y^2 = 3$ parallel to the line $y = 2x + 4$ is

(a) $y = 3x + 4$ (b) $y = 2x + 1$
(c) $y = 2x - 2$ (d) $y = 3x + 5$

3 Let $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$, where $\theta + \phi = \dfrac{\pi}{2}$ be two points on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. If (h, k) is the point of intersection of normals at P and Q, then k is equal to

(a) $\dfrac{a^2+b^2}{a}$ (b) $-\left[\dfrac{a^2+b^2}{a}\right]$ (c) $\dfrac{a^2+b^2}{b}$ (d) $-\left[\dfrac{a^2+b^2}{b}\right]$

4 The equation of the asymptotes of the hyperbola $3x^2 + 4y^2 + 8xy - 8x - 4y - 6 = 0$ is

(a) $3x^2 + 4y^2 + 8xy - 8x - 4y - 3 = 0$
(b) $3x^2 + 4y^2 + 8xy - 8x - 4y + 3 = 0$
(c) $3x^2 + 4y^2 + 8xy - 8x - 4y + 6 = 0$
(d) $4x^2 + 3y^2 + 2xy - x + y + 3 = 0$

5 The angle between the rectangular hyperbolas $(y - mx)(my + x) = a^2$ and $(m^2 - 1)(y^2 - x^2) + 4mxy = b^2$ is

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{4}$ (d) $\dfrac{\pi}{6}$

6 If PQ is a double ordinate of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola. Then, the eccentricity e of the hyperbola satisfies

(a) $1 < e < \dfrac{2}{\sqrt{3}}$ (b) $e = \dfrac{2}{\sqrt{3}}$
(c) $e = \dfrac{\sqrt{3}}{2}$ (d) $e > \dfrac{2}{\sqrt{3}}$

7 The normal at P to a hyperbola of eccentricity e, intersects its transverse and conjugate axes at L and M respectively. If locus of the mid-point of LM is hyperbola, then eccentricity of the hyperbola is

(a) $\left(\dfrac{e+1}{e-1}\right)$ (b) $\dfrac{e}{\sqrt{(e^2-1)}}$ (c) e (d) None of these

8 Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the points P and Q. If these tangents intersect at the point $T(0, 3)$, then the area (in sq units) of ΔPTQ is

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(a) $45\sqrt{5}$ (b) $54\sqrt{3}$
(c) $60\sqrt{3}$ (d) $36\sqrt{5}$

9 If tangents PQ and PR are drawn from variable point P to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 (a > b)$ so that the fourth vertex S of parallelogram $PQSR$ lies on circumcircle of ΔPQR, then locus of P is

(a) $x^2 + y^2 = b^2$ (b) $x^2 + y^2 = a^2$
(c) $x^2 + y^2 = (a^2 - b^2)$ (d) None of these

10 The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is

(a) $3y = 9x + 2$ (b) $y = 2x + 1$
(c) $2y = y + 8$ (d) $y = x + 2$

11 A series of hyperbolas is drawn having a common transverse axis of length $2a$. Then, the locus of a point P on each hyperbola, such that its distance from the transverse axis is equal to its distance from an asymptote, is

(a) $(y^2 - x^2)^2 = 4x^2(x^2 - a^2)$ (b) $(x^2 - y^2)^2 = x^2(x^2 - a^2)$
(c) $(x^2 - y^2) = 4x^2(x^2 - a^2)$ (d) None of these

12 The exhaustive set of values of α^2 such that there exists a tangent to the ellipse $x^2 + \alpha^2 y^2 = \alpha^2$ and the portion of the tangent intercepted by the hyperbola $\alpha^2 x^2 - y^2 = 1$ subtends a right angle at the center of the curves is

(a) $\left[\dfrac{\sqrt{5}+1}{2}, 2\right]$ (b) $(1, 2]$
(c) $\left[\dfrac{\sqrt{5}-1}{2}, 1\right)$ (d) $\left[\dfrac{\sqrt{5}-1}{2}, 1\right) \cup \left(1, \dfrac{\sqrt{5}+1}{2}\right]$

13 A rectangular hyperbola whose centre is C is cut by any circle of radius r in four points P, Q, R and S. Then, $CP^2 + CQ^2 + CR^2 + CS^2$ is equal to

(a) r^2 (b) $2r^2$
(c) $3r^2$ (d) $4r^2$

14 A point P moves such that sum of the slopes of the normals drawn from it to the hyperbola $xy = 4$ is equal to the sum of ordinates of feet of normals. Then, the locus of P is

(a) a parabola (b) a hyperbola
(c) an ellipse (d) a circle

15 A triangle is inscribed in the rectangular hyperbola $xy = c^2$, such that two of its sides are parallel to the lines $y = m_1 x$ and $y = m_2 x$. Then, the third side of the triangle touches the hyperbola

(a) $xy = \left\{\dfrac{c^2(m_1+m_2)^2}{4m_1m_2}\right\}$ (b) $xy = \left\{\dfrac{c^2(m_1-m_2)^2}{4m_1m_2}\right\}$
(c) $xy = \left\{\dfrac{c^2(m_1-m_2)^2}{m_1m_2}\right\}$ (d) $xy = \left\{\dfrac{c^2(m_1+m_2)^2}{m_1m_2}\right\}$

ANSWERS

SESSION 1	**1.** (a)	**2.** (b)	**3.** (d)	**4.** (d)	**5.** (c)	**6.** (a)	**7.** (b)	**8.** (c)	**9.** (a)	**10.** (b)
	11. (c)	**12.** (c)	**13.** (c)	**14.** (b)	**15.** (b)	**16.** (b)	**17.** (d)	**18.** (b)	**19.** (a)	**20.** (a)
	21. (d)	**22.** (a)	**23.** (a)	**24.** (d)	**25.** (c)	**26.** (d)	**27.** (b)	**28.** (b)	**29.** (d)	**30.** (d)
	31. (b)	**32.** (b)	**33.** (a)	**34.** (a)	**35.** (a)					
SESSION 2	**1.** (b)	**2.** (b)	**3.** (d)	**4.** (a)	**5.** (a)	**6.** (d)	**7.** (b)	**8.** (a)	**9.** (c)	**10.** (d)
	11. (a)	**12.** (a)	**13.** (d)	**14.** (a)	**15.** (a)					

Hints and Explanations

SESSION 1

1 Since, the vertices are on the Y-axis (with origin as the mid-point), the equation is of the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

As vertices are $(0, \pm 6)$

$\therefore \quad a=6, b^2=a^2(e^2-1)=36\left(\frac{25}{9}-1\right)$
$=64$

So, the required equation of the hyperbola is $\frac{y^2}{36}-\frac{x^2}{64}=1$ and the foci are $(0, \pm ae)=(0, \pm 10)$.

2 Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \ldots\text{(i)}$$

we have, length of the latusrectum $=8$

$\Rightarrow \quad \frac{2b^2}{a}=8 \Rightarrow b^2=4a$

$\Rightarrow \quad a^2(e^2-1)=4a \Rightarrow a(e^2-1)=4$

$\Rightarrow \quad a\left(\frac{9}{5}-1\right)=4 \Rightarrow a=5$

Putting $a=5$ in $b^2=4a$, we get $b^2=20$

Hence, the equation of the required hyperbola is $\frac{x^2}{25}-\frac{y^2}{20}=1$.

3 Given that the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is passing through the points $(3, 0)$ and $(3\sqrt{2}, 2)$, so we get $a^2=9$ and $b^2=4$.

Again, we know that $b^2=a^2(e^2-1)$.

This gives $4=9(e^2-1)$

$\Rightarrow \quad e^2=\frac{13}{9} \Rightarrow e=\frac{\sqrt{13}}{3}$

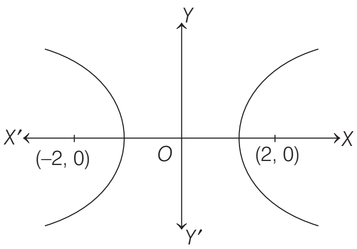
4 Given equation can be rewritten as

$$\frac{y^2}{k^2}-\frac{x^2}{(-k)}=1(-k>0)$$

$$e^2=1+\frac{(-k)}{k^2}=1-\frac{k}{k^2}$$

$\Rightarrow \quad e=\sqrt{1-\frac{1}{k}}$

5



Let equation of hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

where, $\quad 2ae=4$ and $e=2$

$\Rightarrow \quad a=1$

$\because \quad a^2e^2=a^2+b^2 \Rightarrow 4=1+b^2$

$\therefore \quad b^2=3$

Thus, equation of hyperbola is

$$\frac{x^2}{1}-\frac{y^2}{3}=1 \text{ or } 3x^2-y^2=3$$

6 The given equation may be written as

$$\frac{x^2}{\frac{32}{3}}-\frac{y^2}{8}=1 \Rightarrow \frac{x^2}{\left(\frac{4\sqrt{2}}{\sqrt{3}}\right)^2}-\frac{y^2}{(2\sqrt{2})^2}=1$$

On comparing the given equation with the standard equation

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get

$$a^2=\left(\frac{4\sqrt{2}}{\sqrt{3}}\right)^2 \text{ and } b^2=(2\sqrt{2})^2$$

$\therefore$ Length of transverse axis of a hyperbola $=2a=2\times\frac{4\sqrt{2}}{\sqrt{3}}=\frac{8\sqrt{2}}{\sqrt{3}}$

7 Let equation of hyperbola be

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Length of transverse axis is $2a$ and

Length of latusrectum is $\frac{2b^2}{a}$.

Now, difference

$$E=\left|2a-\frac{2b^2}{a}\right|=\frac{2}{a}|2a^2-a^2e^2|$$

$\therefore$ Difference $=2a\,|2-e^2|$

8 Given equation of hyperbola is

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and equation of conjugate hyperbola is $\frac{x^2}{b^2}-\frac{y^2}{a^2}=1$

Since, e and e' are the eccentricities of the respective hyperbola, then

$$e^2=1+\frac{b^2}{a^2}, (e')^2=1+\frac{a^2}{b^2}$$

$$\therefore \quad \frac{1}{e^2}+\frac{1}{e'^2}=\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}=1$$

9 Here, $a=\sin\theta$

Since, foci of the ellipse are $(\pm 1, 0)$.

$\therefore \quad \pm 1=\sqrt{a^2+b^2} \Rightarrow b^2=\cos^2\theta$

Then, equation is $\frac{x^2}{\sin^2\theta}-\frac{y^2}{\cos^2\theta}=1$

$\Rightarrow \quad x^2\operatorname{cosec}^2\theta-y^2\sec^2\theta=1$

10 Given equation can be rewritten as $(x-4)(y-3)=4$ which is a rectangular hyperbola of the type $xy=c^2$.

$\therefore \quad c=2$

Then, $\quad a=b=c\sqrt{2}=2\sqrt{2}$

$\therefore$ Length of latusrectum

$$=\frac{2b^2}{a}=2a=4\sqrt{2}$$

11 Let equation of hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Given, $\frac{2b^2}{a}=8 \Rightarrow \frac{b^2}{a}=4$

and $2b=\frac{1}{2}(2ae) \Rightarrow 2b=ae$

$\Rightarrow \quad 4b^2=a^2e^2 \Rightarrow 4\left(\frac{b^2}{a^2}\right)=e^2$

$\Rightarrow \quad 4(e^2-1)=e^2 \qquad [\because b^2=a^2(e^2-1)]$

$\Rightarrow \quad 3e^2=4 \Rightarrow e^2=\frac{4}{3} \Rightarrow e=\frac{2}{\sqrt{3}}$

12 Now, taking option (c).

Let $x = a\frac{e^t+e^{-t}}{2} \Rightarrow \frac{2x}{a} = e^t+e^{-t}$ ...(i)

and $\frac{2y}{b} = e^t - e^{-t}$ ...(ii)

On squaring and subtracting Eq. (ii) from Eq. (i), we get

$$\frac{4x^2}{a^2}-\frac{4y^2}{b^2}=4 \Rightarrow \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

13 The given equation of hyperbola is

$$\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=1$$

Here, $a^2=\cos^2\alpha$ and $b^2=\sin^2\alpha$

So, the coordinates of foci are $(\pm ae, 0)$.

$\therefore \quad e = \sqrt{1+\frac{b^2}{a^2}} \Rightarrow e = \sqrt{1+\frac{\sin^2\alpha}{\cos^2\alpha}}$

$\Rightarrow \quad e=\sqrt{1+\tan^2\alpha}=\sec\alpha$

Hence, abscissae of foci remain constant when α varies.

14 Let equation of tangent to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $y = mx+\sqrt{a^2m^2-b^2}$

i.e. $\quad mx - y + \sqrt{a^2m^2-b^2}=0$

$\therefore$ Required product

$= \left|\frac{mae+\sqrt{a^2m^2-b^2}}{\sqrt{m^2+1}}\right| \left|\frac{-mae+\sqrt{a^2m^2-b^2}}{\sqrt{m^2+1}}\right|$

$=\left|\frac{a^2m^2-b^2-m^2a^2e^2}{m^2+1}\right|$

$=\left|\frac{m^2a^2(1-e^2)-b^2}{m^2+1}\right| = \left|\frac{-m^2b^2-b^2}{m^2+1}\right|$

[$\because b^2=a^2(e^2-1)$]

$=b^2$

15. Here, $a^2=\frac{232}{7}, b^2=\frac{232}{5}$

and $y=-\frac{21}{5}x+\frac{116}{5}$ with slope $-\frac{21}{5}$.

Now, $a^2m^2-b^2=\left(\frac{116}{5}\right)^2$

[since, line is tangent]

If (x_1,y_1) is the point of contact, then tangent is $S_1=0$

$\therefore \quad 7x_1x-5y_1y=232$

On comparing it with $21x+5y=116$, we get $x_1=6, y_1=-2$

So, the point of contact is $(6,-2)$.

16 Here, $e_1=\sqrt{1-\frac{16}{25}}=\frac35$

$\because \quad e_1e_2=1 \Rightarrow e_2=\frac53$

Since, foci of ellipse are $(0,\pm3)$.

Hence, equation of hyperbola is

$\frac{x^2}{16}-\frac{y^2}{9}=-1$.

17 As we know, equation of tangent at (x_1,y_1) is $xx_1-2yy_1=4$, which is same as $2x+\sqrt6 y=2$

$\therefore \quad \frac{x_1}{2}=-\frac{2y_1}{\sqrt6}=\frac42 \Rightarrow x_1=4$

and $\quad y_1=-\sqrt6$

18 Let the equation of hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

$\therefore \quad ae=2\Rightarrow a^2e^2=4$

$\Rightarrow \quad a^2+b^2=4 \Rightarrow b^2=4-a^2$

$\therefore \quad \frac{x^2}{a^2}-\frac{y^2}{4-a^2}=1$

Since, $(\sqrt2,\sqrt3)$ lie on hyperbola.

$\therefore \quad \frac{2}{a^2}-\frac{3}{4-a^2}=1$

$\Rightarrow \quad 8-2a^2-3a^2=a^2(4-a^2)$

$\Rightarrow \quad 8-5a^2=4a^2-a^4$

$\Rightarrow \quad a^4-9a^2+8=0$

$\Rightarrow \quad (a^2-8)(a^2-1)=0$

$\Rightarrow \quad a^2=8, a^2=1$

$\therefore \quad a=1$

Now, equation of hyperbola is $\frac{x^2}{1}-\frac{y^2}{3}=1$.

$\therefore$ Equation of tangent at $(\sqrt2,\sqrt3)$ is given by

$\sqrt2x-\frac{\sqrt3y}{3}=1 \Rightarrow \sqrt2x-\frac{y}{\sqrt3}=1$

which passes through point $(2\sqrt2, 3\sqrt3)$.

19 Suppose the common tangent is $y=mx+c$ to

$9x^2-4y^2=36$ and $x^2+y^2=3$

$\therefore \quad c^2=a^2m^2-b^2=4m^2-9$ ...(i)

and $c^2=3+3m^2$ ...(ii)

From Eqs. (i) and (ii), we get

$4m^2-9=3m^2+3\Rightarrow m^2=12$

$\Rightarrow \quad m=2\sqrt3$

$\therefore \quad c=\sqrt{3+3\times12}=\sqrt{39}$

Hence, the common tangent is $y=2\sqrt3x+\sqrt{39}$

20 Since, the line $y=\alpha x+\beta$ is tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

$\therefore \quad \beta^2=a^2\alpha^2-b^2$.

So, locus of (α,β) is $y^2=a^2x^2-b^2$

$\Rightarrow \quad a^2x^2-y^2-b^2=0$

Since, this equation represents a hyperbola, so locus of a point $P(\alpha,\beta)$ is a hyperbola.

21 Given equation of hyperbola is $\frac{x^2}{9}-\frac{y^2}{4}=1$.

Here, $\quad a^2=9, b^2=4$

and equation of line is

$$y=-x+\sqrt2p \quad \text{...(i)}$$

If the line $y=mx+c$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $c^2=a^2m^2-b^2$ ...(ii)

From Eq. (i), we get $m=-1, c=\sqrt2p$

On putting these values in Eq. (ii), we get

$(\sqrt2p)^2=9(1)-4\Rightarrow 2p^2=5$

22 We know that the locus of the point of intersection of perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is a circle $x^2+y^2=a^2-b^2$

Thus, locus of the point of intersection of perpendicular tangents to the hyperbola $\frac{x^2}{3}-\frac{y^2}{1}=1$ is a circle

$x^2+y^2=3-1\Rightarrow x^2+y^2=2$

23 The point on the hyperbola is $P(x_1,y_1)$, then N is $(x, 0)$.

$\therefore$ Tangent at (x_1,y_1) is $\frac{xx_1}{9}-\frac{yy_1}{4}=1$

This meets X-axis at $T\left(\frac{9}{x_1},0\right)$

$\therefore \quad OT\cdot ON=\frac{9}{x_1}\cdot x_1=9$

24 The equation of tangent in slope form to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ is

$y=mx+\sqrt{16m^2-9}$

Since, it passes through (h, k).

$\therefore \quad k=mh+\sqrt{16m^2-9^2}$

$\Rightarrow \quad (k-mh)^2=(16m^2-9^2)$

$\Rightarrow k^2+m^2h^2-2mkh-16m^2+9^2=0$

It is quadratic in m and let the slope of two tangents be m_1 and m_2, then

$m_1m_2=\frac{k^2+9}{h^2-16}$

$\Rightarrow \quad -1=\frac{k^2+9}{h^2-16}\Rightarrow h^2+k^2=7$

The required locus is $x^2+y^2=7$

25 Any point on the hyperbola is $(3\sec\theta, 2\tan\theta)$. The chord of contact to the circle is $3x\sec\theta+2y\tan\theta=9$ ...(i)

If (x_1,y_1) is the mid-point of the chord, then its equation is

$xx_1+yy_1=x_1^2+y_1^2$ ...(ii)

From Eqs. (i) and (ii), we get

$\frac{3\sec\theta}{x_1}=\frac{2\tan\theta}{y_1}=\frac{9}{x_1^2+y_1^2}$

Eliminating θ, $\frac{1}{81}(x_1^2+y_1^2)^2=\frac{x_1^2}{9}-\frac{y_1^2}{4}$

Hence, the locus is

$(x^2+y^2)^2=81\left(\frac{x^2}{9}-\frac{y^2}{4}\right)$

26 Given hyperbola is $\frac{x^2}{4}-\frac{y^2}{2}=1$.

Here, $a^2=4, b^2=2 \Rightarrow a=2, b=\sqrt{2}$

The equation of tangent is

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$$

$$\Rightarrow \frac{x}{2}\sec\theta-\frac{y}{\sqrt{2}}\tan\theta=1$$

So, the coordinates of P and Q are $P(2\cos\theta, 0)$ and $Q(0, -\sqrt{2}\cot\theta)$, respectively.

Let coordinates of R are (h, k).

$\therefore \quad h=2\cos\theta, k=-\sqrt{2}\cot\theta$

$\Rightarrow \frac{k}{h}=\frac{-\sqrt{2}}{\sin\theta} \Rightarrow \sin\theta=\frac{-\sqrt{2}h}{2k}$

On squaring both sides, we get

$$\sin^2\theta=\frac{2h^2}{4k^2} \Rightarrow 1-\cos^2\theta=\frac{2h^2}{4k^2}$$

$$\Rightarrow 1-\frac{h^2}{4}=\frac{2h^2}{4k^2} \Rightarrow \frac{2h^2}{4k^2}+\frac{h^2}{4}=1$$

$$\Rightarrow \frac{h^2}{4}\left(\frac{2}{k^2}+1\right)=1 \Rightarrow \frac{2}{k^2}+1=\frac{4}{h^2}$$

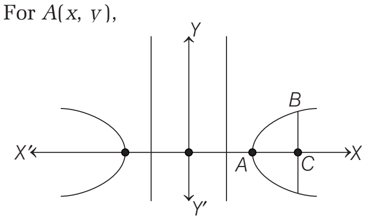
$$\Rightarrow \frac{4}{h^2}-\frac{2}{k^2}=1$$

Hence, R lies on $\frac{4}{x^2}-\frac{2}{y^2}=1$.

27 The given equation can be rewritten as

$$\frac{(x-\sqrt{2})^2}{4}-\frac{(y+\sqrt{2})^2}{2}=1$$

For $A(x, y)$,



$$e=\sqrt{1+\frac{2}{4}}=\sqrt{\frac{3}{2}}$$

$\therefore \quad x-\sqrt{2}=2 \Rightarrow x=2+\sqrt{2}$

For $C(x, y)$, $x-\sqrt{2}=ae=\sqrt{6}$

$\therefore \quad x=\sqrt{6}+\sqrt{2}$

Now, $AC=\sqrt{6}+\sqrt{2}-2-\sqrt{2}=\sqrt{6}-2$

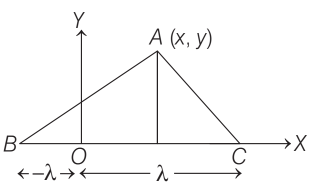
and $BC=\frac{b^2}{a}=\frac{2}{2}=1$

$\therefore$ Area of $\Delta ABC=\frac{1}{2}\times AC\times BC$

$$=\frac{1}{2}\times(\sqrt{6}-2)\times 1=\left(\sqrt{\frac{3}{2}}-1\right) \text{ sq unit}$$

28 Let $B(-\lambda, 0), C(\lambda, 0)$ and $A(x, y)$.

Given, $K=\angle B-\angle C$



$\therefore \quad \tan K=\frac{\tan B-\tan C}{1+\tan B\cdot\tan C}$

$$=\frac{\frac{y}{\lambda+x}-\frac{y}{\lambda-x}}{1+\frac{y^2}{\lambda^2-x^2}}$$

$\Rightarrow \quad \lambda^2-x^2+y^2=-2xy\cot K$

$\Rightarrow \quad x^2-2\cot K\cdot xy-y^2=\lambda^2$

which is a hyperbola.

29 Equation of normal at θ is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$ and normal at $\phi=\frac{\pi}{2}-\theta$ is $\frac{ax}{\operatorname{cosec}\theta}+\frac{by}{\cot\theta}=a^2+b^2$

Eliminating x, we get

$$by\left(\frac{1}{\sin\theta}-\frac{1}{\cos\theta}\right)=(a^2+b^2)\left(\frac{1}{\cos\theta}-\frac{1}{\sin\theta}\right)$$

$\Rightarrow \quad by=-(a^2+b^2)$ or $k=-\frac{(a^2+b^2)}{b}$

30 The equation of hyperbola is

$4x^2-9y^2=36 \Rightarrow \frac{x^2}{9}-\frac{y^2}{4}=1 \quad$ ...(i)

The equation of the chords of contact of tangents from (x_1, y_1) and (x_2, y_2) to the given hyperbola are

$\frac{xx_1}{9}-\frac{yy_1}{4}=1 \quad$ ...(ii)

and $\frac{xx_2}{9}-\frac{yy_2}{4}=1 \quad$ ...(iii)

Lines (ii) and (iii) are at right angles.

$$\frac{9}{4}\cdot\frac{x_1}{y_1}\times\frac{4}{9}\cdot\frac{x_2}{y_2}=-1 \Rightarrow \frac{x_1x_2}{y_1y_2}=-\left(\frac{9}{4}\right)^2=-\frac{81}{16}$$

31 Let (h, k) be the point whose chord of contact w.r.t. hyperbola $x^2-y^2=9$ is $x=9$. We know that chord of (h, k) w.r.t. hyperbola $x^2-y^2=9$ is $T=0$

$\Rightarrow \quad hx-ky-9=0$

But it is the equation of line $x=9$. This is possible only when $h=1, k=0$. Again, equation of pair of tangents is $T^2=SS_1$

$\Rightarrow \quad (x-9)^2=(x^2-y^2-9)(1-9)$

$\Rightarrow \quad x^2-18x+81=(x^2-y^2-9)(-8)$

$\Rightarrow \quad 9x^2-8y^2-18x+9=0$

32 Equation of chord of hyperbola $x^2-y^2=a^2$ with mid-point as (h, k) is given by

$$xh-yk=h^2-k^2 \Rightarrow y=\frac{h}{k}x-\frac{(h^2-k^2)}{k}$$

This will touch the parabola $y^2=4ax$, if

$$-\left(\frac{h^2-k^2}{k}\right)=\frac{a}{h/k} \Rightarrow ak^2=-h^3+k^2h$$

$\therefore$ Locus of the mid-point is

$x^3=y^2(x-a)$

33 Let (h, k) is mid-point of chord.

Then, its equation is

$$3hx-2ky+2(x+h)-3(y+k)=3h^2-2k^2+4h-6k$$

$\Rightarrow \quad x(3h+2)+y(-2k-3)=3h^2-2k^2+2h-3k$

Since, this line is parallel to $y=2x$.

$\therefore \quad \frac{3h+2}{2k+3}=2 \Rightarrow 3h+2=4k+6$

$\Rightarrow \quad 3h-4k=4$

Thus, locus of mid-point is $3x-4y=4$

34 The mid-point of the chord is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

The equation of the chord $T=S_1$.

$$\therefore \quad x\left(\frac{y_1+y_2}{2}\right)+y\left(\frac{x_1+x_2}{2}\right)=2\left(\frac{x_1+x_2}{2}\right)\left(\frac{y_1+y_2}{2}\right)$$

$\Rightarrow \quad x(y_1+y_2)+y(x_1+x_2)=(x_1+x_2)(y_1+y_2)$

$\Rightarrow \quad \frac{x}{x_1+x_2}+\frac{y}{y_1+y_2}=1$

35 **A.** Since, vertices $(\pm 2, 0)$ and foci $(\pm 3, 0)$ lie on X-axis, as coefficient of Y-axis is zero.

Hence, equation of hyperbola will be of the form

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad$ ...(i)

Here, it is given that $a=2$ and $c=3$

$\because \quad c^2=a^2+b^2$

$\Rightarrow \quad 9=4+b^2 \Rightarrow b^2=5$

Put the values of $a^2=4$ and $b^2=5$ in Eq. (i), we get

$\frac{x^2}{4}-\frac{y^2}{5}=1 \quad$ ...(ii)

B. Since, vertices $(0, \pm 5)$ and foci $(0, \pm 8)$ lie on Y-axis as coefficient of X-axis is zero. Hence, equation of hyperbola will be of the form

$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1 \quad$ ...(i)

Here, it is given that

$(0, \pm a)=(0, \pm 5)$ and foci

$(0, \pm c)=(0, \pm 8)$

$\Rightarrow \quad a=5$ and $c=8$

$\because \quad c^2=a^2+b^2 \Rightarrow b^2=39$

Put $a^2=25$ and $b^2=39$ in Eq. (i), we get

$$\frac{y^2}{25}-\frac{x^2}{39}=1$$

C. Since, vertices $(0, \pm 3)$ and foci $(0, \pm 5)$ lie on Y- axis as coordinate of x is zero.

Hence, equation of hyperbola will be of the form

$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1 \quad$ ...(i)

Here, it is given that vertices $(0, \pm 3) = (0, \pm a)$ and foci

$$(0, \pm 5) = (0, \pm c)$$

$\Rightarrow \quad a = 3$ and $c = 5$

$\because \quad c^2 = a^2 + b^2$

$\Rightarrow \quad 25 = 9 + b^2 \Rightarrow b^2 = 16$

Put $a^2 = 9$ and $b^2 = 16$ in Eq. (i), we get

$$\frac{y^2}{9} - \frac{x^2}{16} = 1$$

SESSION 2

1 If (h, k) is the mid-point of the chord, then its equation by $T = S_1$ is

$$hx - ky = h^2 - k^2$$

$$\Rightarrow \quad y = \frac{h}{k}x + \frac{k^2 - h^2}{k}$$

If it touches the parabola $y^2 = 4ax$, then we get

$$\frac{k^2 - h^2}{k} = \frac{a \cdot k}{h} \Rightarrow ak^2 = hk^2 - h^3$$

$$\Rightarrow \quad ay^2 = xy^2 - x^3$$

So, required locus is $y^2(x - a) = x^3$.

2 Equation of the hyperbola is

$$3x^2 - y^2 = 3 \Rightarrow \frac{x^2}{1} - \frac{y^2}{3} = 1$$

Equation of tangent in terms of slope

$$y = mx \pm \sqrt{(m^2 - 3)}$$

Given, $\quad m = 2$

$\therefore \quad y = 2x \pm 1$

3 Equation of the tangents at $P(a\sec\theta, b\tan\theta)$ is $\dfrac{x}{a}\sec\theta - \dfrac{y}{b}\tan\theta = 1$

$\therefore$ Equation of the normal at P is

$$ax + b\operatorname{cosec}\theta\, y = (a^2 + b^2)\sec\theta \quad \text{...(i)}$$

Similarly, the equation of normal at $Q(a\sec\phi, b\tan\phi)$ is

$$ax + b\operatorname{cosec}\phi\, y = (a^2 + b^2)\sec\phi \quad \text{...(ii)}$$

On subtracting Eq. (ii) from Eq. (i), we get

$$y = \frac{a^2 + b^2}{b} \cdot \frac{\sec\theta - \sec\phi}{\operatorname{cosec}\theta - \operatorname{cosec}\phi}$$

So, $\quad k = y = \dfrac{a^2 + b^2}{b} \cdot \dfrac{\sec\theta - \sec\left(\dfrac{\pi}{2} - \theta\right)}{\operatorname{cosec}\theta - \operatorname{cosec}\left(\dfrac{\pi}{2} - \theta\right)}$

$$= \frac{a^2 + b^2}{b} \cdot \frac{\sec\theta - \operatorname{cosec}\theta}{\operatorname{cosec}\theta - \sec\theta} = -\left[\frac{a^2 + b^2}{b}\right]$$

4 The equation of the asymptotes of the hyperbola

$$3x^2 + 4y^2 + 8xy - 8x - 4y - 6 = 0$$

is $\quad 3x^2 + 4y^2 + 8xy - 8x - 4y + \lambda = 0$

It should represent a pair of straight lines.

$\therefore \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$3 \cdot 4 \cdot \lambda + 2 \cdot (-2)(-4)\, 4 - 3(-2)^2$$

$$-4(-4)^2 - \lambda(4)^2 = 0$$

$\Rightarrow \quad 12\lambda + 64 - 12 - 64 - 16\lambda = 0$

$\Rightarrow \quad -4\lambda - 12 = 0 \Rightarrow \lambda = -3$

$\therefore$ Required equation is

$$3x^2 + 4y^2 + 8xy - 8x - 4y - 3 = 0$$

5 Given equations are

$$(y - mx)(my + x) = a^2 \quad \text{...(i)}$$

and $\quad (m^2 - 1)(y^2 - x^2) + 4mxy = b^2 \quad \text{...(ii)}$

On differentiating Eq.(i), we get

$$(y - mx)\left(m\frac{dy}{dx} + 1\right) + (my + x)\left(\frac{dy}{dx} - m\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx}(my + x + my - m^2x) + y - mx - m^2y - mx = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y + m^2y + 2mx}{2my + x - m^2x} = m_1 \quad \text{[say]}$$

On differentiating Eq.(ii), we get

$$(m^2 - 1)\left(2y\frac{dy}{dx} - 2x\right) + 4m\left(x\frac{dy}{dx} + y\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx}[2y(m^2 - 1) + 4mx] = -4my + 2x(m^2 - 1)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2my + m^2x - x}{m^2y - y + 2mx} = m_2 \quad \text{[say]}$$

$\because \quad m_1 m_2 = -1$

So, angle between the hyperbola $= \dfrac{\pi}{2}$.

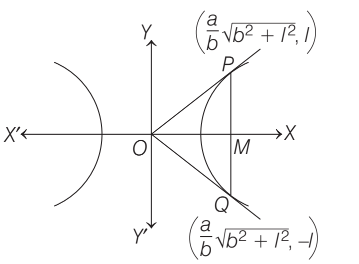
6 $\because$ PQ is the double ordinate. Let $MP = MQ = l$.

Given that ΔOPQ is an equilateral, then $OP = OQ = PQ$

$\Rightarrow \quad (OP)^2 = (OQ)^2 = (PQ)^2$

$$\Rightarrow \frac{a^2}{b^2}(b^2 + l^2) + l^2 = \frac{a^2}{b^2}(b^2 + l^2) + l^2 = 4l^2$$

$$\Rightarrow \quad \frac{a^2}{b^2}(b^2 + l^2) = 3l^2$$



$$\Rightarrow \quad a^2 = l^2\left(3 - \frac{a^2}{b^2}\right)$$

$$\Rightarrow \quad l^2 = \frac{a^2b^2}{(3b^2 - a^2)} > 0$$

$\therefore \quad 3b^2 - a^2 > 0$

$\Rightarrow \quad 3b^2 > a^2$

$\Rightarrow \quad 3a^2(e^2 - 1) > a^2$

$\Rightarrow \quad e^2 > 4/3 \therefore e > \dfrac{2}{\sqrt{3}}$

7 Equation of normal at $P(a\sec\phi, b\tan\phi)$ is $a\,x\cos\phi + by\cot\phi = a^2 + b^2$.

Then, coordinates of L and M are $\left(\dfrac{a^2 + b^2}{a} \cdot \sec\phi, 0\right)$ and $\left(0, \dfrac{a^2 + b^2}{b}\tan\phi\right)$ respectively.

Let mid-point of ML is $Q(h, k)$,

then $\quad h = \dfrac{(a^2 + b^2)}{2a}\sec\phi$

$$\therefore \quad \sec\phi = \frac{2ah}{(a^2 + b^2)} \quad \text{...(i)}$$

and $\quad k = \dfrac{(a^2 + b^2)}{2b}\tan\phi$

$$\therefore \quad \tan\phi = \frac{2bk}{(a^2 + b^2)} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\sec^2\phi - \tan^2\phi = \frac{4a^2h^2}{(a^2 + b^2)^2} - \frac{4b^2k^2}{(a^2 + b^2)^2}$$

Hence, required locus is

$$\frac{x^2}{\left(\dfrac{a^2 + b^2}{2a}\right)^2} - \frac{y^2}{\left(\dfrac{a^2 + b^2}{2b}\right)^2} = 1$$

Let eccentricity of this curve is e_1.

$$\Rightarrow \quad \left(\frac{a^2 + b^2}{2b}\right)^2 = \left(\frac{a^2 + b^2}{2a}\right)^2(e_1^2 - 1)$$

$$\Rightarrow a^2 = b^2(e_1^2 - 1) \Rightarrow a^2 = a^2(e^2 - 1)(e_1^2 - 1)$$

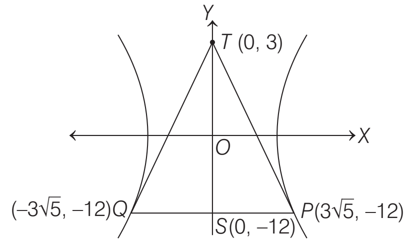
$$[\because b^2 = a^2(e^2 - 1)]$$

$\Rightarrow \quad e^2e_1^2 - e^2 - e_1^2 + 1 = 1$

$\Rightarrow \quad e_1^2(e^2 - 1) = e^2 \Rightarrow e_1 = \dfrac{e}{\sqrt{(e^2 - 1)}}$

8 Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the point P and Q.

Tangent intersects at point $T(0, 3)$



Clearly, PQ is chord of contact.

$\therefore$ Equation of PQ is $-3y = 36$

$\Rightarrow \quad y = -12$

Solving the curve $4x^2 - y^2 = 36$ and $y = -12$, we get $x = \pm 3\sqrt{5}$

Area of $\quad \Delta PQT = \dfrac{1}{2} \times PQ \times ST$

$$= \frac{1}{2}(6\sqrt{5} \times 15) = 45\sqrt{5}$$

9 Fourth vertex of parallelogram lies on circumcircle

$\Rightarrow$ Parallelogram is cyclic.

$\Rightarrow$ Parallelogram is a rectangle.

$\Rightarrow$ Tangents are perpendicular

$\Rightarrow$ Locus of P is the director circle

i.e., $\quad x^2 + y^2 = a^2 - b^2$

10 Any point on parabola $y^2 = 8x$ is $(2t^2, 4t)$. The equation of tangent at that point is

$$yt = x + 2t^2 \qquad \text{...(i)}$$

Given that, $xy = -1 \qquad$ ...(ii)

On solving Eqs. (i) and (ii), we get

$y(yt - 2t^2) = -1 \Rightarrow ty^2 - 2t^2 y + 1 = 0$

$\because$ It is common tangent. It means they are intersect only at one point and the value of discriminant is equal to zero.

i.e. $\quad 4t^4 - 4t = 0 \Rightarrow t = 0, 1$

$\therefore$ The common tangent is $y = x + 2$, (when $t = 0$, it is $x = 0$ which can touch $xy = -1$ at infinity only)

11 The equation of a hyperbola of the series is $\frac{x^2}{a^2} - \frac{y^2}{\lambda^2} = 1$ where, λ is a parameter. The asymptotes of this hyperbola $\frac{x}{a} = \pm \frac{y}{\lambda}$. Suppose (x', y') is a point P on the hyperbola which is equidistant from the transverse axis and asymptote.

Then, $\quad \frac{x'^2}{a^2} - \frac{y'^2}{\lambda^2} = 1 \qquad$ ...(i)

and $\quad y' = \dfrac{x'/a - y'/\lambda}{\sqrt{\frac{1}{a^2} + \frac{1}{\lambda^2}}} \qquad$ ...(ii)

i.e. $\quad \frac{y'^2}{\lambda^2} = \frac{x'^2}{a^2} - 1$ [from Eq. (i)] ...(iii)

and $y'^2 \left(\frac{1}{a^2} + \frac{1}{\lambda^2}\right) = \frac{x'^2}{a^2} + \frac{y'^2}{\lambda^2} - \frac{2x'y'}{a\lambda}$

[from Eq. (ii)] ...(iv)

On simplification the second relation gives

$$(y'^2 - x'^2)^2 = \frac{4x'^2 y'^2 a^2}{\lambda^2}$$

$$= 4x'^2 (x'^2 - a^2) \text{ [using Eq. (iii)]}$$

So, the locus of P is

$$(y^2 - x^2)^2 = 4x^2 (x^2 - a^2).$$

12 The equation of tangent at point $P(\alpha\cos\theta, \sin\theta)$

$$\frac{x}{\alpha}\cos\theta + \frac{y}{1}\sin\theta = 1 \qquad \text{...(i)}$$

Let it cut the hyperbola at points P and Q. Homogenising hyperbola $\alpha^2 x^2 - y^2 = 1$ with the help of Eq. (i), we get

$$\alpha^2 x^2 - y^2 = \left(\frac{x}{\alpha}\cos\theta + y\sin\theta\right)^2$$

This is a pair of straight lines $OP\ OQ$.

Given $\angle POQ = \pi/2$.

Coefficient of x^2 + Coefficient of $y^2 = 0$

or $\quad \alpha^2 - \frac{\cos^2\theta}{\alpha^2} - 1 - \sin^2\theta = 0$

or $\quad \alpha^2 - \frac{\cos^2\theta}{\alpha^2} - 1 - 1 + \cos^2\theta = 0$

or $\quad \cos^2\theta = \frac{\alpha^2(2 - \alpha^2)}{\alpha^2 - 1}$

Now, $0 \le \cos^2\theta \le 1$

or $\quad 0 \le \frac{\alpha^2(2 - \alpha^2)}{\alpha^2 - 1} \le 1$

After solving, we get $\alpha^2 \in \left[\frac{\sqrt{5} + 1}{2}, 2\right]$

13 Let equation of the rectangular hyperbola be $xy = c^2 \qquad$ ...(i)

and equation of circle be

$$x^2 + y^2 = r^2. \qquad \text{...(ii)}$$

From Eq. (i) and Eq.(ii) eliminating y, we get

$$x^4 - r^2 x^2 + c^4 = 0 \qquad \text{...(iii)}$$

Let x_1, x_2, x_3 and x_4 are the roots of Eq. (iii).

$\therefore$ Sum of roots $= \sum_{i=1}^{4} x_i = 0$

Sum of products of the roots taken two at a time $= \Sigma x_1 x_2 = -r^2$

From Eq. (i) and Eq. (ii) eliminating x, we get

$$y^4 - r^2 y^2 + c^4 = 0 \qquad \text{...(iv)}$$

Let y_1, y_2, y_3 and y_4 are the roots of Eq. (iv).

$\therefore \quad \sum_{i=1}^{4} y_i = 0$ and $\Sigma y_1 y_2 = -r^2$

Now, $CP^2 + CQ^2 + CR^2 + CS^2$

$$= x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2 + x_4^2 + y_4^2$$

$$= (x_1^2 + x_2^2 + x_3^2 + x_4^2) + (y_1^2 + y_2^2 + y_3^2 + y_4^2)$$

$$= \left[\left(\sum_{i=1}^{4} x_i\right)^2 - 2\Sigma x_1 x_2\right] + \left[\left(\sum_{i=1}^{4} y_i\right)^2 - 2\Sigma y_1 y_2\right]$$

$$= (0 + 2r^2) + (0 + 2r^2)$$

$\left[\because \left(\sum_{i=1}^{4} x_i\right) = 0, \Sigma x_1 x_2 = -r^2, \sum_{i=1}^{4} y_i = 0, \Sigma y_1 y_2 = -r^2\right]$

$$= 4r^2$$

14 Any point on the hyperbola $xy = 4$ is $\left(2t, \frac{2}{t}\right)$. Now, normal at $\left(2t, \frac{2}{t}\right)$ is

$y - \frac{2}{t} = t^2 (x - 2t)$. $\qquad$ [its slope is t^2]

If the normal passes through $P(h, k)$, then

$$k - \frac{2}{t} = t^2 (h - 2t)$$

$\Rightarrow \quad 2t^4 - ht^3 + tk - 2 = 0 \qquad$ ...(i)

Roots of Eq. (i) give parameters of feet of normals passing through (h, k). Let roots be t_1, t_2, t_3 and t_4, then

$$t_1 + t_2 + t_3 + t_4 = \frac{h}{2} \qquad \text{...(ii)}$$

$$t_1 t_2 + t_2 t_3 + t_1 t_3 + t_1 t_4 + t_2 t_4 + t_3 t_4 = 0 \qquad \text{...(iii)}$$

$$t_1 t_2 t_3 + t_1 t_2 t_4 + t_1 t_3 t_4 + t_2 t_3 t_4 = -\frac{k}{2} \qquad \text{...(iv)}$$

and $\quad t_1 t_2 t_3 t_4 = -1 \qquad$ ...(v)

On dividing Eq. (iv) by Eq. (v), we get

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = \frac{k}{2}$$

$\Rightarrow \quad y_1 + y_2 + y_3 + y_4 = k \quad \left[\because y = \frac{2}{t}\right]$

From Eqs. (ii) and (iii), we get

$$t_1^2 + t_2^2 + t_3^2 + t_4^2 = \frac{h^2}{4}$$

Given that, $\quad \frac{h^2}{4} = k$

Hence, locus of (h, k) is $x^2 = 4ay$, which is a parabola.

15 Let $P\left(ct_1, \frac{c}{t_1}\right)$, $Q\left(ct_2, \frac{c}{t_2}\right)$, $R\left(ct_3, \frac{c}{t_3}\right)$ be the vertices of a ΔPQR inscribed in the rectangular hyperbola $xy = c^2$ such that the sides PQ and QR are parallel to $y = m_1 x$ and $y = m_2 x$, respectively.

$\therefore \quad m_1$ = Slope of PQ

and m_2 = Slope of QR

$\Rightarrow \quad m_1 = -\frac{1}{t_1 t_2}$

and $\quad m_2 = -\frac{1}{t_2 t_3}$

$\therefore \quad \frac{m_1}{m_2} = \frac{t_3}{t_1} \Rightarrow t_3 = \left(\frac{m_1}{m_2}\right) t_1$

The equation of PR is

$$x + y t_1 t_3 = c(t_1 + t_3)$$

$\Rightarrow \quad x + y\left(\frac{m_1}{m_2}\right) t_1^2 = c\left(t_1 + \frac{m_1}{m_2} t_1\right)$

$\Rightarrow \quad x + y\left(\frac{m_1}{m_2}\right) t_1^2 = c\left(1 + \frac{m_1}{m_2}\right) t_1$

$\Rightarrow \quad x + y\left(\sqrt{\frac{m_1}{m_2}}\, t_1\right)^2 = 2\left\{\frac{c(m_1 + m_2)}{2\sqrt{m_1 m_2}} \sqrt{\frac{m_1}{m_2}} \cdot t_1\right\}$

$\Rightarrow \quad x + yt^2 = 2\lambda t$, where,

$$t = \sqrt{\frac{m_1}{m_2}} \cdot t_1 \text{ and } \lambda = \frac{c(m_1 + m_2)}{2\sqrt{m_1 m_2}}$$

Clearly, it touches the hyperbola,

$$xy = \left\{\frac{c(m_1 + m_2)}{2\sqrt{m_1 m_2}}\right\}^2$$

or $\quad xy = \left\{\frac{c^2 (m_1 + m_2)^2}{4 m_1 m_2}\right\}$